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(A Dynamic Problem)

By

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ON THE DETERMINATION OF VERTICAL SETTLING OF THE BOUNDARY OF A HALF SPACE IN THE CASE OF PLANAR DEFORMATION
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I. WE USE AN ORTHOGONAL SYSTEM OF COORDINATES (x,y,z) THE y-AXIS OF WHICH IS PERPENDICULAR TO THE BOUNDARY OF A HALF-SPACE.

Lame's equations for the elastic isotropic half-space in the planar case have, in the absence of volumetric forces, the following form:

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\nabla^2 u=\rho\frac{\partial^2 u}{\partial t^2},\\(\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\nabla^2 v=\rho\frac{\partial^2 v}{\partial t^2},\end{aligned}\right\}\qquad(1.1)$$

where λ, μ are Lame's constants of elasticity; u, ν are the projections of the displacement vector on the axes of the coordinates x, y;

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y};$$

p is the density of the material; and

$$\nabla^2=\frac{\partial^2}{dx^2}+\frac{\partial^2}{\partial y^2}.$$

The stress components are connected with displacement components by correlations for planar deformations which appear as:

$$\sigma_x=\lambda\theta+2\mu\frac{\partial u}{\partial x};$$

$$\sigma_y=\lambda\theta+2\mu\frac{\partial v}{\partial y};$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

where

$$\overline{F}(\alpha, y, p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} l^{-pt+i\alpha x} \Phi(x, y, t) dt.$$

The solution of equation (1.8) which is bounded at infinity appears as:

$$\overline{F}(\alpha, y, p) = A exp\left(-y\sqrt{\alpha^2 + \frac{p^2}{c_1^2}}\right) + B exp\left(-y\sqrt{\alpha^2 + \frac{p^2}{c_2^2}}\right), \quad (1.9)$$

where A and B are determined from conditions on the boundary of the half-space.

Successively applying to (1.3), (1.6) and (1.7) a Laplace transform with reference to the variable t, and a Fourier transform with reference to the variable x, we shall have the following relationship between the Laplace-Fourier transformants of the functions ν (x, y, t), σ_y (x, y, t), τ_{xy}(x, y, t), and of the function $\bar{F}$ (α, y, p). Substituting into these relationships functions $\bar{F}$ (a, y, p) from (1.9), and also assuming that y = 0, we obtain the following expression for the Laplace-Fourier transformant of the functions ν (x, 0, t), σ_y(x, 0, t) and τ_{xy} (x, 0, t):

$$\overline{V}(\alpha, 0, p) = -\frac{\lambda + \mu}{\mu}\left[\left(\alpha^2 + \frac{p^2}{c_1^2}\right)A + \alpha^2 B\right], \quad (1.10)$$

$$\overline{\Sigma}_y(\alpha, 0, p) = 2(\lambda + \mu)\left[\left(\alpha^2 + \frac{p^2}{2c_2^2}\right)\left(\alpha^2 + \frac{p^2}{c_1^2}\right)^{1/2} A + \alpha^2\left(\alpha^2 + \frac{p^2}{c_2^2}\right)^{1/2} B\right], \quad (1.11)$$

$$\overline{T}_{xy}(\alpha, 0, p) = 2(\lambda + \mu) i\alpha\left[\left(\alpha^2 + \frac{p^2}{c_1^2}\right) A + \left(\alpha^2 + \frac{p^2}{2c_2^2}\right) B\right]. \quad (1.12)$$

II. OUR PROBLEM CONSISTS IN DETERMINING THE FUNCTIONS ν (x, 0, t) IF σ_y(x, 0, t) AND τ_{xy}(x, 0, t) ARE KNOWN.

We obtain the solution of this problem by utilizing the results of paragraph I of this article. In fact, when the functions σ_y(x, 0, t) and τ_{xy}(x, 0, t) are known, then the Laplace-Fourier Transformants of the functions $\overline{\Sigma}_y$ (α, 0, p) and $\overline{T}_{xy}$ (α, 0, p), generally speaking, also may be

considered as known. On the basis of this, from (1.11) and (1.12) expressions for A (a, p) and B (a, p) can easily be found. Substituting the expressions for A (a, p) and B (a, p) into (1.10) and successively applying the inversion theorem for the Laplace and Fourier transform /2/, we can obtain the formula for the determination of the function ν (x, 0, t).

Let us examine in more detail the case τ_{xy}(x, 0, t) = 0. Substituting the expressions A (a, p) and B (a, p) from (1.11) and (1.12) for the case τ_{xy} (x, 0, t) = 0 in (1.10) after algebraic transformations, we obtain the following formula for the Laplace-Fourier transforms of the function ν (x, 0, t):

$$\bar{v}(\alpha, 0, p) = -\frac{4c_2^6}{\mu}\frac{\Sigma_y(\alpha, 0, p)}{\prod_{k=1}^{6}(p - i\alpha c_2 x_k)}\left[\alpha^2\left(\alpha^2 + \frac{p^2}{c_1^2}\right)\left(\alpha^2 + \frac{p^2}{c_2^2}\right)^{1/2} + \right.$$
$$\left. + \left(\alpha^2 + \frac{p^2}{2c_2^2}\right)^2\left(\alpha^2 + \frac{p^2}{c_1^2}\right)^{1/2}\right], \qquad (2.1)$$

where x_k are the roots of the equation

$$x^6 - 8x^4 + \left(24 - 16\frac{c_2^2}{c_1^2}\right)x^2 + \left(16\frac{c_2^2}{c_1^2} - 16\right) = 0. \qquad (2.2)$$

The roots of equation (2.3) depend on the relation

$$\frac{c_2^2}{c_1} = \frac{\mu}{\lambda + 2\mu} = \frac{1 - 2\nu}{2 - 2\nu},$$

where ν is the Poisson's ratio.

Successively applying the theorems of inversion to (2.1) for the Laplace and Fourier transforms, we obtain:

$$v(x, 0, t) = -\frac{4c_2^6}{\mu}\frac{1}{(2\pi)^{3/2}i}\int_{-\infty}^{\infty}d\alpha\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt - i\alpha x}\frac{\Sigma_y(\alpha, 0, p)}{\prod_{k=1}^{6}(p - i\alpha c_2 x_k)} \times$$
$$\times\left[\alpha^2\left(\alpha^2 + \frac{p^2}{c_1^2}\right)\left(\alpha^2 + \frac{p^2}{c_2^2}\right)^{1/2} + \left(\alpha^2 + \frac{p^2}{2c_2^2}\right)^2\left(\alpha^2 + \frac{p^2}{c_1^2}\right)^{1/2}\right]dp. \qquad (2.3)$$

The final stage of solving the present problem is the calculation of the contour integrals entering into expression (2.3). However, for practical use it is better to have a formula for function v (x, 0, t) into which real integrals - not contour integrals - enter. For obtaining such a formula, we apply successively to (2.4) the theorems about the convolutions for the Laplace transform in reference to the variable t, and for the Fourier transform in reference to the variable x /2/. As a result we get:

$$v(x,0,t)=\frac{2}{\pi\mu}\int_0^t d\tau\int_{-\infty}^{\infty}\Psi(\xi,\tau)\frac{\partial}{\partial\tau}\sigma_y(x-\xi,0,t-\tau)\,d\xi, \quad (2.4)$$

where

$$\Psi(\xi,\tau)=\begin{cases} 0 \quad \text{at } 0\leqslant\tau<\frac{|\xi|}{c_1}, \\ A_0f_1+A_1(\varphi_1-\pi)+A_2(\varphi_2-\pi)+A_3(\varphi_3+i\pi) \\ \quad \text{at } \frac{|\xi|}{c_1}<\tau<\frac{|\xi|}{c_2}, \\ A_0(f_1+f_2)+A_1(\varphi_1-\psi_1)+A_2(\varphi_2-\psi_2)+A_3(\varphi_3+\psi_3+2i\pi) \\ \quad \text{at } \frac{|\xi|}{c_2}<\tau<\frac{|\xi|}{\varkappa_3c_2}, \\ A_0(f_1+f_2)+A_1(\varphi_1-\psi_1)+A_2(\varphi_2-\psi_2)+A_3(\varphi_3+\psi_3) \\ \quad \text{at } \tau<\frac{|\xi|}{\varkappa_3c_2}, \end{cases} \quad (2.5)$$

$$\left.\begin{aligned} A_0&=\frac{1-\nu}{4};\quad A_1=\frac{\left(\varkappa_1^2\frac{c_2^2}{c_1^2}-1\right)^{1/2}\left(1-\frac{1}{2}\varkappa_1^2\right)^2}{\varkappa_1^2\left(\varkappa_2^2-\varkappa_1^2\right)\left(\varkappa_3^2-\varkappa_1^2\right)};\\ A_2&=\frac{\left(\varkappa_2^2\frac{c_2^2}{c_1^2}-1\right)^{1/2}\left(1-\frac{1}{2}\varkappa_2^2\right)^2}{\varkappa_2^2\left(\varkappa_3^2-\varkappa_2^2\right)\left(\varkappa_1^2-\varkappa_2^2\right)};\quad A_3=\frac{\left(1-\varkappa_3^2\frac{c_2^2}{c_1^2}\right)^{1/2}\left(1-\frac{1}{2}\varkappa_3^2\right)^2}{\varkappa_3^2\left(\varkappa_1^2-\varkappa_3^2\right)\left(\varkappa_2^2-\varkappa_3^2\right)}, \end{aligned}\right\} \quad (2.6)$$

$$\left.\begin{aligned} f_1(\xi,\tau)&=\ln\frac{c_1\tau+\sqrt{c_1^2\tau^2-\xi^2}}{|\xi|},\\ f_2(\xi,\tau)&=\ln\frac{c_2\tau+\sqrt{c_2^2\tau^2-\xi^2}}{|\xi|}, \end{aligned}\right\} \quad (2.7)$$

$$\left.\begin{aligned}
\varphi_1(\xi,\tau) &= \frac{\pi}{2} + \arcsin\frac{c_1\tau - \varkappa_1\frac{c_2}{c_1}|\xi|}{|\xi| - \varkappa_1 c_2\tau},\\
\varphi_2(\xi,\tau) &= \frac{\pi}{2} + \arcsin\frac{c_1\tau - \varkappa_2\frac{c_2}{c_1}|\xi|}{|\xi| - \varkappa_2 c_2\tau},\\
\varphi_3(\xi,\tau) &= \ln\frac{c_1\tau - \varkappa_3\frac{c_2}{c_1}|\xi| - \sqrt{\left(1-\chi_3^2\frac{c_2^2}{c_1^2}\right)\left(c_1^2\tau^2-\xi^2\right)}}{\varkappa_3 c_2\tau - |\xi|},\\
\psi_1(\xi,\tau) &= \frac{\pi}{2} + \arcsin\frac{c_2\tau - \varkappa_1|\xi|}{|\xi| - \varkappa_1 c_2\tau},\\
\psi_2(\xi,\tau) &= \frac{\pi}{2} + \arcsin\frac{c_2\tau - \varkappa_2|\xi|}{|\xi| - \varkappa_2 c_2\tau},\\
\psi_3(\xi,\tau) &= \ln\frac{c_2\tau - \varkappa_3|\xi| - \sqrt{\left(1-\varkappa_3^2\right)\left(c_2^2\tau^2-\xi^2\right)}}{\varkappa_3 c_2\tau - |\xi|}.
\end{aligned}\right\} \quad (2.8)$$

Thus, a general solution to the problem of determining the vertical components of the displacement vector of the boundary of the half-space in a planar case is reached by formula (2.4) when $t > 0$ stress acts on the boundary of the half-space at $\sigma_y\,(x, 0, t)$ and $\tau_{xy}(x, 0, t) = 0$.

The obtained solution is valid when the following conditions are fulfilled:

a. At $t \geqslant 0$ the half-space is in the state of rest;

b. $\sigma_y\,(x, 0, 0) = 0$;

c. All roots of the equation (2.2) are simple and real;

d. When the inequalities are valid $0 < \chi_3 c_2 < c_2 < c_1 \leq \chi_2 c_2 < \chi_1 c_2$,

where γ_1, χ_2, χ_3 are the positive roots of the equation (2.2). For instance, at $\gamma = 0.25$ conditions в and г are fulfilled. However, in case of necessity this method will give a corresponding solution, also, when the above enumerated conditions are valid.

III. IN THE FOLLOWING IS SHOWN AN APPLICATION OF THE GENERAL FORMULA (2.4) FOR THE SOLUTION OF INDIVIDUAL PROBLEMS.

Problem a. Let

$$\sigma_y(x, 0, t) = -P\delta(x) H(t), \tag{3.1}$$

where $\delta(x)$ is the Dirac delta-function, and $H(t)$ is a discontinuous function determined by the correlation

$$Ht = \begin{cases} 0 & \text{at } t < 0, \\ 1 & \text{at } t > 0. \end{cases}$$

The case under examination is graphically depicted in Fig. 1.

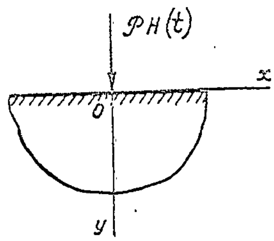

Fig. 1

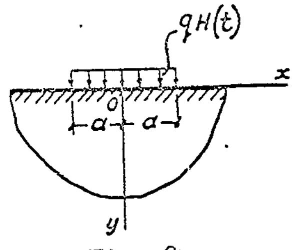

Fig. 2

In this case

$$\frac{\partial}{\partial t}\sigma_y(x, 0, t) = -P\delta(x)\delta(t)$$

and formula (2.4) gives

$$v(x, 0, t) = \frac{2P}{\pi\mu}\int_0^t d\tau \int_{-\infty}^{\infty} \Psi(\xi, \tau)\delta(t-\tau)\delta(x-\xi)\, d\xi.$$

Utilizing the properties of the delta-function /4/ we finally get

$$v(x, 0, t) = \frac{2P}{\pi\mu}\Psi(x, t), \tag{3.2}$$

where the function ψ is determined by the formulas (2.5)-(2.8), wherein the variables ζ and τ are substituted by x and t respectively. When the boundary

value is calculated by the function v (x, 0, t) determined by the expression (3.2) at $t \to \infty$, then we arrive at a known formula

$$\lim_{t \to \infty} v(x, 0, t) = \frac{2P(1-\nu^2)}{\pi E} \ln \frac{|d|}{|x|}, \qquad (3.3)$$

which is obtained from the investigations of Flamant /5/, and which corresponds to the case of a statical application of stresses σ_y (x, 0) = -P δ (x). Here the expression E = 2 (1 + v) μ is the modulus of the normal (perpendicular) stress. It must be noted that formula (3.3) does not determine the absolute value of the vertical component of the displacement vector of any arbitrary point located on the boundary of the half-space, but the difference of the vertical components of the displacement vector at the point with the abscissa (x) and a certain point also located on the boundary but removed at a distance of (d) from the origin of the coordinates.

Problem b. Let

$$\sigma_y(x, 0, t) = -q H(a - |x|) H(t). \qquad (3.4)$$

The examined case is graphically depicted in Fig. 2. In the given case

$$\frac{\partial}{\partial t} \sigma_y(x, 0, t) = -q H(d - |x|) \delta(t)$$

and formula (2.5) gives

$$v(x, 0, t) = \frac{2q}{\pi\mu} \int_0^t d\tau \int_{x-a}^{x+a} \Psi(\xi, \tau) \delta(t - \tau) d\xi.$$

Changing the order of integration in this expression and accomplishing the integration by τ , we get

$$v(x, 0, t) = \frac{2q}{\pi\mu} \int_{x-a}^{x+a} \Psi(\xi, t) d\xi. \qquad (3.5)$$

Integration in formula (3.5) can be accomplished without undue difficulties. But since the final expression in this case for function ν (x, 0, t) is too cumbersome, we will not write it out.

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